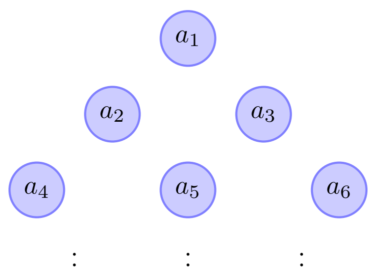
BALTIC WAY 2015
Problems and Solutions

Problem 1.
For $n \geq 2$, an equilateral triangle is divided into n^2 congruent smaller equilateral triangles. Determine all ways in which real numbers can be assigned to the $\frac{(n+1)(n+2)}{2}$ vertices so that three such numbers sum to zero whenever the three vertices form a triangle with edges parallel to the sides of the big triangle.

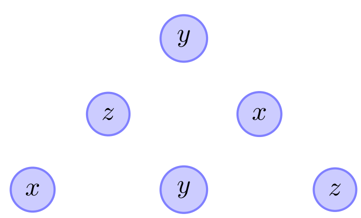
Solution. We label the vertices (and the corresponding real numbers) as follows.



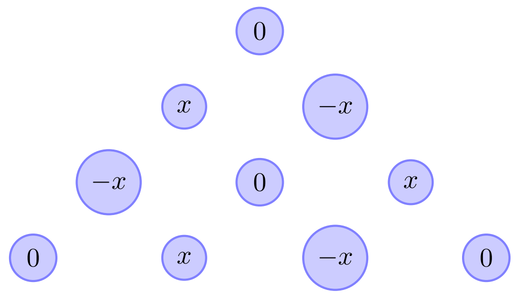
For $n = 2$, we see that

$$a_2 + a_4 + a_5 = 0 = a_2 + a_3 + a_5,$$

which shows that $a_3 = a_4$ and similarly $a_1 = a_5$ and $a_2 = a_6$. Now the only additional requirement is $a_1 + a_2 + a_3 = 0$, so that all solutions are of the following form, for any x, y and z with $x + y + z = 0$:



For $n = 3$, observe that $a_1 = a_7 = a_{10}$ since they all equal a_5. Since also $a_1 + a_7 + a_{10} = 0$, they all equal zero. By considering the top triangle, we get $x = a_2 = -a_3$ and this uniquely determines the rest. It is easily checked that, for any real x, this is actually a solution:



For $n \geq 3$ we can apply the same argument as above for any collection of 10 vertices. Any vertex not on the sides of the big triangle has to equal zero, since it is the centre of such a collection of 10 vertices. Any vertex a on the sides of the big triangle forms some parallelogram similar to a_4, a_2, a_5, a_8, where the point opposite a is in the interior of the big triangle. Since such opposite numbers are equal, all a_i have to be zero in this case. □

Problem 2.
Let n be a positive integer and let $a_1, \ldots, a_n$ be real numbers satisfying $0 \leq a_i \leq 1$ for $i = 1, \ldots, n$. Prove the inequality

$$(1 - a_1^n)(1 - a_2^n) \cdots (1 - a_n^n) \leq (1 - a_1 a_2 \cdots a_n)^n.$$

Solution. The numbers $1 - a_i^n$ are positive by assumption. AM–GM gives

$$\begin{aligned}(1 - a_1^n)(1 - a_2^n) \cdots (1 - a_n^n) \leq & \left(\frac{(1 - a_1^n) + (1 - a_2^n) + \cdots + (1 - a_n^n)}{n} \right)^n \\ = & \left(1 - \frac{a_1^n + \cdots + a_n^n}{n} \right)^n.\end{aligned}$$

By applying AM–GM again we obtain

$$a_1 a_2 \cdots a_n \leq \frac{a_1^n + \cdots + a_n^n}{n} \quad \Rightarrow \quad \left(1 - \frac{a_1^n + \cdots + a_n^n}{n} \right)^n \leq (1 - a_1 a_2 \cdots a_n)^n,$$

and hence the desired inequality. □

Remark. It is possible to use Jensen's inequality applied to $f(x) = \log(1 - e^x)$.

Problem 3.
Let $n > 1$ be an integer. Find all non-constant real polynomials $P(x)$ satisfying, for any real x, the identity

$$P(x)P(x^2)P(x^3) \cdots P(x^n) = P\left(x^{\frac{n(n+1)}{2}}\right).$$

Solution. Answer: $P(x) = x^m$ if n is even; $P(x) = \pm x^m$ if n is odd.

Consider first the case of a monomial $P(x) = ax^m$ with $a \neq 0$. Then

$$ax^{\frac{mn(n+1)}{2}} = P\left(x^{\frac{n(n+1)}{2}}\right) = P(x)P(x^2)P(x^3) \cdots P(x^n) = ax^m \cdot ax^{2m} \cdot \cdots \cdot ax^{nm} = a^n x^{\frac{mn(n+1)}{2}}$$

implies $a^n = a$. Thus, $a = 1$ when n is even and $a = \pm 1$ when n is odd. Obviously these polynomials satisfy the desired equality.

Suppose now that P is not a monomial. Write $P(x) = ax^m + Q(x)$, where Q is non-zero polynomial with $\deg Q = k < m$. We have

$$\begin{aligned} ax^{\frac{mn(n+1)}{2}} + Q\left(x^{\frac{n(n+1)}{2}}\right) &= P\left(x^{\frac{n(n+1)}{2}}\right) \\ &= P(x)P(x^2)P(x^3)\cdots P(x^n) = (ax^m + Q(x))(ax^{2m} + Q(x^2))\cdots(ax^{nm} + Q(x^n)). \end{aligned}$$

The highest degree of a monomial, on both sides of the equality, is $\frac{mn(n+1)}{2}$. The second highest degree in the right-hand side is

$$2m + 3m + \cdots + nm + k = \frac{m(n+2)(n-1)}{2} + k,$$

while in the left-hand side it is $\frac{kn(n+1)}{2}$. Thus

$$\frac{m(n+2)(n-1)}{2} + k = \frac{kn(n+1)}{2},$$

which leads to

$$(m-k)(n+2)(n-1) = 0,$$

and so $m = k$, contradicting the assumption that $m > k$. Consequently, no polynomial of the form $ax^m + Q(x)$ fulfils the given condition. □

Problem 4.
A family wears clothes of three colours: red, blue and green, with a separate, identical laundry bin for each colour. At the beginning of the first week, all bins are empty. Each week, the family generates a total of 10 kg of laundry (the proportion of each colour is subject to variation). The laundry is sorted by colour and placed in the bins. Next, the heaviest bin (only one of them, if there are several that are heaviest) is emptied and its contents washed. What is the minimal possible storing capacity required of the laundry bins in order for them never to overflow?

Solution. Answer: 25 kg.

Each week, the accumulation of laundry increases the total amount by $K = 10$, after which the washing decreases it by at least one third, because, by the pigeon-hole principle, the bin with the most laundry must contain at least a third of the total. Hence the amount of laundry post-wash after the nth week is bounded above by the sequence $a_{n+1} = \frac{2}{3}(a_n + K)$ with $a_0 = 0$, which is clearly bounded above by $2K$. The total amount of laundry is less than $2K$ post-wash and $3K$ pre-wash.

Now suppose pre-wash state (a, b, c) precedes post-wash state $(a, b, 0)$, which precedes pre-wash state (a', b', c'). The relations $a \leq c$ and $a' \leq a + K$ lead to

$$3K > a + b + c \geq 2a \geq 2(a' - K),$$

and similarly for b', whence $a', b' < \frac{5}{2}K$. Since also $c' \leq K$, a pre-wash bin, and *a fortiori* a post-wash bin, always contains less than $\frac{5}{2}K$.

Consider now the following scenario. For a start, we keep packing the three bins equally full before washing. Initialising at $(0, 0, 0)$, the first week will end at $(\frac{1}{3}K, \frac{1}{3}K, \frac{1}{3}K)$ pre-wash and $(\frac{1}{3}K, \frac{1}{3}K, 0)$ post-wash, the second week at $(\frac{5}{9}K, \frac{5}{9}K, \frac{5}{9}K)$ pre-wash and $(\frac{5}{9}K, \frac{5}{9}K, 0)$ post-wash, &c. Following this scheme, we can get arbitrarily close to the state $(K, K, 0)$ after washing. Supposing this accomplished, placing $\frac{1}{2}K$ kg of laundry in each of the non-empty bins leaves us in a state close to $(\frac{3}{2}K, \frac{3}{2}K, 0)$ pre-wash and $(\frac{3}{2}K, 0, 0)$ post-wash. Finally, the next week's worth of laundry is directed solely to the single non-empty bin. It may thus contain any amount of laundry below $\frac{5}{2}K$ kg. □

Problem 5.
Find all functions $f\colon \mathbf{R} \to \mathbf{R}$ satisfying the equation

$$|x|f(y) + yf(x) = f(xy) + f(x^2) + f(f(y))$$

for all real numbers x and y.

Solution. Answer: all functions $f(x) = c(|x| - x)$, where $c \geq 0$.

Choosing $x = y = 0$, we find

$$f(f(0)) = -2f(0).$$

Denote $a = f(0)$, so that $f(a) = -2a$, and choose $y = 0$ in the initial equation:

$$a|x| = a + f(x^2) + f(a) = a + f(x^2) - 2a \quad \Rightarrow \quad f(x^2) = a(|x| + 1).$$

In particular, $f(1) = 2a$. Choose $(x, y) = (z^2, 1)$ in the initial equation:

$$\begin{aligned} & z^2 f(1) + f(z^2) = f(z^2) + f(z^4) + f(f(1)) \\ \Rightarrow \quad & 2az^2 = z^2 f(1) = f(z^4) + f(f(1)) = a(z^2 + 1) + f(2a) \\ \Rightarrow \quad & az^2 = a + f(2a). \end{aligned}$$

The right-hand side is constant, while the left-hand side is a quadratic function in z, which can only happen if $a = 0$. (Choose $z = 1$ and then $z = 0$.)

We now conclude that $f(x^2) = 0$, and so $f(x) = 0$ for all non-negative x. In particular, $f(0) = 0$. Choosing $x = 0$ in the initial equation, we find $f(f(y)) = 0$ for all y. Simplifying the original equation and swapping x and y leads to

$$|x|f(y) + yf(x) = f(xy) = |y|f(x) + xf(y).$$

Choose $y = -1$ and put $c = \frac{f(-1)}{2}$:

$$|x|f(-1) - f(x) = f(x) + xf(-1) \quad \Rightarrow \quad f(x) = \frac{f(-1)}{2}(|x| - x) = c(|x| - x).$$

One easily verifies that these functions satisfy the functional equation for any parameter $c \geq 0$. □

Problem 6.
Two players take alternate turns in the following game. At the outset there are two piles, containing 10,000 and 20,000 tokens, respectively. A move consists of removing any positive number of tokens from a single pile *or* removing $x > 0$ tokens from one pile and $y > 0$ tokens from the other, where $x + y$ is divisible by 2015. The player who cannot make a move loses. Which player has a winning strategy?

Solution. The first player wins.

He should present his opponent with one of the following positions:

$$(0, 0), \quad (1, 1), \quad (2, 2), \quad \ldots, \quad (2014, 2014).$$

All these positions have different total numbers of tokens modulo 2015. Therefore, if the game starts from two piles of arbitrary sizes, it is possible to obtain one of these positions just by the first move. In our case

$$10{,}000 + 20{,}000 \equiv 1790 \bmod 2015,$$

and the first player can leave to his opponent the position $(895, 895)$.

Now the second type of move can no longer be carried out. If the second player removes n tokens from one pile, the first player may always respond be removing n tokens from the other pile. □

Problem 7.
There are 100 members in a ladies' club. Each lady has had tea (in private) with exactly 56 of the other members of the club. The Board, consisting of the 50 most distinguished ladies, have all had tea with one another. Prove that the entire club may be split into two groups in such a way that, within each group, any lady has had tea with any other.

Solution. Each lady in the Board has had tea with 49 ladies within the Board, and 7 ladies without. Each lady not in the Board has had tea with at most 49 ladies not in the Board, and at least 7 ladies in the Board. Comparing these two observations, we conclude that each lady not in the Board has had tea with exactly 49 ladies not in the Board and exactly 7 ladies in the Board. Hence the club may be split into Board members and non-members. □

Problem 8.
With inspiration drawn from the rectilinear network of streets in New York, the *Manhattan distance* between two points (a, b) and (c, d) in the plane is defined to be

$$|a - c| + |b - d|.$$

Suppose only two distinct Manhattan distances occur between all pairs of distinct points of some point set. What is the maximal number of points in such a set?

Solution. Answer: nine.

Let

$$\{(x_1, y_1), \ldots, (x_m, y_m)\}, \qquad \text{where} \qquad x_1 \leq \cdots \leq x_m,$$

be the set, and suppose $m \geq 10$.

A special case of the Erdős–Szekeres Theorem asserts that a real sequence of length $n^2 + 1$ contains a monotonic subsequence of length $n + 1$. (Proof: Given a sequence $a_1 \ldots, a_{n^2+1}$, let p_i denote the length of the longest increasing subsequence ending with a_i, and q_i the length of the longest decreasing subsequence ending with a_i. If $i < j$ and $a_i \leq a_j$, then $p_i < p_j$. If $a_i \geq a_j$, then $q_i < q_j$. Hence all $n^2 + 1$ pairs (p_i, q_i) are distinct. If all of them were to satisfy $1 \leq p_i, q_i \leq n$, it would violate the Pigeon-Hole Principle.)

Applied to the sequence $y_1, \ldots, y_m$, this will produce a subsequence

$$y_i \leq y_j \leq y_k \leq y_l \qquad \text{or} \qquad y_i \geq y_j \geq y_k \geq y_l.$$

One of the shortest paths from (x_i, y_i) to (x_l, y_l) will pass through first (x_j, y_j) and then (x_k, y_k). At least three distinct Manhattan distances will occur.

Conversely, among the nine points

$$(0, 0), \qquad (\pm 1, \pm 1), \qquad (\pm 2, 0), \qquad (0, \pm 2),$$

only the Manhattan distances 2 and 4 occur. □

Problem 9.
Let $n > 2$ be an integer. A deck contains $\frac{n(n-1)}{2}$ cards, numbered

$$1,\ 2,\ 3,\ \ldots,\ \frac{n(n-1)}{2}.$$

Two cards form a *magic pair* if their numbers are consecutive, or if their numbers are 1 and $\frac{n(n-1)}{2}$.

For which n is it possible to distribute the cards into n stacks in such a manner that, among the cards in any two stacks, there is exactly one magic pair?

Solution 1. Answer: for all odd n.

First assume a stack contains two cards that form a magic pair; say cards number i and $i+1$. Among the cards in this stack and the stack with card number $i+2$ (they might be identical), there are two magic pairs — a contradiction. Hence no stack contains a magic pair.

Each card forms a magic pair with exactly two other cards. Hence if n is even, each stack must contain at least $\lceil \frac{n-1}{2} \rceil = \frac{n}{2}$ cards, since there are $n-1$ other stacks. But then we need at least $n\frac{n}{2} > \frac{n(n-1)}{2}$ cards — a contradiction.

In the odd case we distribute the cards like this: Let $a_1, a_2, \ldots, a_n$ be the n stacks and let $n = 2m+1$. Card number 1 is put into stack a_1. If card number $km+i$, for $i = 1, 2, \ldots, m$, is put into stack a_j, then card number $km+i+1$ is put into stack number a_{j+i}, where the indices are calculated modulo n.

There are
$$\frac{n(n-1)}{2} = \frac{(2m+1)(2m)}{2} = m(2m+1)$$
cards. If we look at all the card numbers of the form $km+1$, there are exactly $n = 2m+1$ of these, and we claim that there is exactly one in each stack. Card number 1 is in stack a_1, and card number $km+1$ is in stack
$$a_{1+k(1+2+3+\cdots+m)}.$$
Since
$$1+2+3+\cdots+m = \frac{m(m+1)}{2}$$
and $\gcd(2m+1, \frac{m(m+1)}{2}) = 1$, all the indices
$$1+k(1+2+3+\cdots+m), \qquad k = 0, 1, 2, \ldots, 2m$$
are different modulo $n = 2m+1$. In the same way we see that each stack contains exactly one of the $2m+1$ cards with the numbers $km+i$ for a given $i = 2, 3, \ldots, m$.

Now look at two different stacks a_v and a_u. Then, without loss of generality, we may assume that $u = v+i$ for some $i = 1, 2, \ldots, m$ (again we consider the index modulo $n = 2m+1$). Since there is a card in stack a_v with number $km+i$, the card $km+i+1$ is in stack $a_{v+i} = a_u$. Hence among the cards in any two stacks there is at least one magic pair. Since there is the same number of pairs of stacks as of magic pairs, there must be exactly one magic pair among the cards of any two stacks. □

Solution 2 (found by Saint Petersburg). For the case of n odd, consider the complete graph on the vertices $1, \ldots, n$ with $\frac{n(n-1)}{2}$ edges. The degree of each vertex is $n-1$, which is even, hence an Euler cycle $v_1 v_2 \cdots v_{\frac{n(n-1)}{2}} v_1$ exists. Place card number i into stack number v_i. The magic pairs correspond to edges in the cycle. □

Problem 10.

A subset S of $\{1, 2, \ldots, n\}$ is called *balanced* if for every $a \in S$ there exists some $b \in S$, $b \neq a$, such that $\frac{a+b}{2} \in S$ as well.

(a) Let $k > 1$ be an integer and let $n = 2^k$. Show that every subset S of $\{1, 2, \ldots, n\}$ with $|S| > \frac{3n}{4}$ is balanced.

(b) Does there exist an $n = 2^k$, with $k > 1$ an integer, for which every subset S of $\{1, 2, \ldots, n\}$ with $|S| > \frac{2n}{3}$ is balanced?

Solution of part (a). Let $m = n - |S|$, thus $m < \frac{n}{4}$ and (as n is a multiple of 4) $m \le \frac{n}{4} - 1$. Let $a \in S$. There are $\frac{n}{2} - 1$ elements in $\{1, 2, \ldots, n\}$ distinct from a and with the same parity as a. At most m of those elements are not in S, hence at least $\frac{n}{2} - 1 - m \ge \frac{n}{4}$ of them are in S. For each such b, the number $\frac{a+b}{2}$ is an integer, and all of these at least $\frac{n}{4}$ numbers are distinct. But at most $m < \frac{n}{4}$ of them are not in S, so at least one is a member of S. Hence S is balanced. □

Solution 1 of part (b). For convenience we work with $\{0, 1, \ldots, n-1\}$ rather than $\{1, 2, \ldots, n\}$; this does not change the problem. We show that one can always find an unbalanced subset containing more than $\frac{2n}{3}$ elements.

Let $\operatorname{ord}_2(i)$ denote the number of factors 2 occurring in the prime factorisation of i. We set

$$T_j = \{\, i \in \{1, 2, \ldots, n-1\} \mid \operatorname{ord}_2(i) = j \,\}.$$

Then we choose

$$S = \{0, 1, 2, \ldots, n-1\} \setminus (T_1 \cup T_3 \cup \cdots \cup T_l), \qquad \text{where } l = \begin{cases} k-1 & \text{if } k \text{ even,} \\ k-2 & \text{if } k \text{ odd.} \end{cases}$$

Observe that $|T_j| = \frac{n}{2^{j+1}}$, so

$$|S| = n - \left(\frac{n}{4} + \frac{n}{16} + \cdots + \frac{n}{2^{l+1}}\right) = n - n \cdot \frac{\frac{1}{4} - \frac{1}{2^{l+3}}}{1 - \frac{1}{4}} > n - \frac{n}{3} = \frac{2n}{3}.$$

We show that S is not balanced. Take $a = 0 \in S$, and consider a $0 \ne b \in S$. If b is odd, then $\frac{0+b}{2}$ is not integral. If b is even, then $b \in T_2 \cup T_4 \cup \cdots$, so $\frac{b}{2} \in T_1 \cup T_3 \cup \cdots$, hence $\frac{b}{2} \notin S$. Thus S is not balanced. □

Solution 2 of part (b). We define the sets

$$A_j = \{2^{j-1} + 1, 2^{j-1} + 2, \ldots, 2^j\},$$

and set

$$S = A_k \cup A_{k-2} \cup \cdots \cup A_l \cup \{1\}, \qquad \text{where } l = \begin{cases} 2 & \text{if } k \text{ even,} \\ 1 & \text{if } k \text{ odd.} \end{cases}$$

Note that $A_j \subseteq \{1, 2, \ldots, n\}$ whenever $j \le k$, and that $|A_j| = 2^{j-1}$. We find

$$|S| = 2^{k-1} + 2^{k-3} + \cdots + 2^{l-1} + 1 = \frac{2^{l-1} - 2^{k+1}}{1-4} + 1 = -\frac{2^{l-1}}{3} + \frac{2n}{3} + 1 > \frac{2n}{3}.$$

We show that S is not balanced. Take $a = 1 \in S$, and consider a $1 \ne b \in S$. Then $b \in A_j$ for some j. If b is even, then $\frac{1+b}{2}$ is not integral. If b is odd, then also $1 + b \in A_j$, so $\frac{1+b}{2} \in A_{j-1}$ and does not lie in S. Thus S is not balanced. □

Solution 3 of part (b). Let us introduce the concept of *lonely element* as an $a \in S$ for which there does not exist a $b \in S$, distinct from a, such that $\frac{a+b}{2} \in S$.

We will construct an unbalanced set S with $|S| > \frac{2n}{3}$ for all k. For $n = 4$ we can use $S = \{1, 2, 4\}$ (all elements are lonely), and for $n = 8$ we can use $S = \{1, 2, 3, 5, 6, 7\}$ (2 and 6 are lonely).

We now construct an unbalanced set $S \subseteq \{1, 2, \ldots, 4n\}$, given an unbalanced set $T \subseteq \{1, 2, \ldots, n\}$ with $|T| > \frac{2n}{3}$. Take

$$S = \{\, i \in \{1, 2, \ldots, 4n\} \mid i \equiv 1 \mod 2 \,\} \cup \{\, 4t - 2 \mid t \in T \,\}.$$

Then

$$|S| = 2n + |T| > 2n + \frac{2n}{3} = \frac{8n}{3} = \frac{2 \cdot 4n}{3}.$$

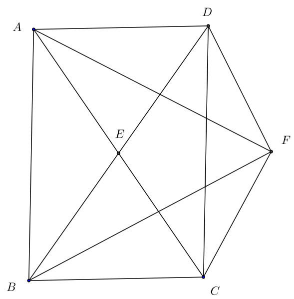


Figure 1: Problem 11.

Supposing $a \in T$ is lonely, we will show that $4a - 2 \in S$ is lonely. Indeed, suppose $4a - 2 \neq b \in S$ with

$$\frac{4a - 2 + b}{2} = 2a - 1 + \frac{b}{2} \in S.$$

Then b must be even, so $b = 4t - 2$ for some $a \neq t \in T$. But then

$$\frac{4a - 2 + 4t - 2}{2} = 4\frac{a + t}{2} - 2,$$

again an even element. However, as a is lonely we know that $\frac{a+t}{2} \notin T$, and hence $4\frac{a+t}{2} - 2 \notin S$. We conclude that $4a - 2$ is lonely in S.

Thus S is an unbalanced set, and by induction we can find an unbalanced set of size exceeding $\frac{2n}{3}$ for all $k > 1$. □

Problem 11.
The diagonals of the parallelogram $ABCD$ intersect at E. The bisectors of $\angle DAE$ and $\angle EBC$ intersect at F. Assume that $ECFD$ is a parallelogram. Determine the ratio $AB : AD$.

Solution. Since $ECFD$ is a parallelogram, we have $ED \parallel CF$ and $\angle CFB = \angle EBF = \angle FBC$ (BF bisects $\angle DBC$). So CFB is an isosceles triangle and $BC = CF = ED$ ($ECFD$ is a parallelogram). In a similar manner, $EC = AD$. But since $ABCD$ is a parallelogram, $AD = BC$, whence $EC = ED$. So the diagonals of $ABCD$ are equal, which means that $ABCD$ is in fact a rectangle. Also, the triangles EDA and EBC are equilateral, and so AB is twice the altitude of EDA, or $AB = \sqrt{3} \cdot AD$. □

Problem 12.
A circle passes through vertex B of the triangle ABC, intersects its sides AB and BC at points K and L, respectively, and touches the side AC at its midpoint M. The point N on the arc BL (which does not contain K) is such that $\angle LKN = \angle ACB$. Find $\angle BAC$ given that the triangle CKN is equilateral.

Solution. Answer: $\angle BAC = 75°$.

Since $\angle ACB = \angle LKN = \angle LBN$, the lines AC and BN are parallel. Hence $ACNB$ is a trapezium. Moreover, $ACNB$ is an isosceles trapezium, because the segment AC touches the circle s in the midpoint (and so the trapezium is symmetrical with respect to the perpendicular bisectors of BN).

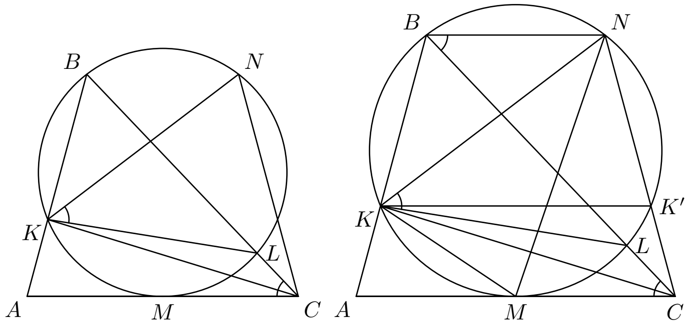


Figure 2: Problem 12.

Denote by K' the intersection point of s and CN. Then the line KK' is parallel to the bases of the trapezium. Hence M is the midpoint of arc KK' and the line NM is an angle bisector of the equilateral triangle KNC.

Thus we obtain that $MC = MK$. Therefore the length of median KM of the triangle AKC equals $\frac{1}{2}AC$; hence $\angle AKC = 90°$. We have

$$2\angle A = \angle KAC + \angle ACN = \angle KAC + \angle ACK + \angle KCN = 90° + 60° = 150°,$$

and so $\angle A = 75°$. □

Problem 13.
Let D be the footpoint of the altitude from B in the triangle ABC, where $AB = 1$. The incentre of triangle BCD coincides with the centroid of triangle ABC. Find the lengths of AC and BC.

Solution. Answer: $AC = BC = \sqrt{\frac{5}{2}}$.

The centroid of ABC lies on the median CC'. It will also, by the assumption, lie on the angle bisector through C. Since the median and the angle bisector coincide, ABC is isosceles with $AC = BC = a$.

Furthermore, the centroid lies on the median BB' and the bisector of $\angle DBC$, again by hypothesis. By the Angle Bisector Theorem,

$$\frac{B'D}{BD} = \frac{B'C}{BC} = \frac{a/2}{a} = \frac{1}{2}.$$

The triangles $ABD \sim ACC'$ since they have equal angles, whence

$$\frac{1}{a} = \frac{AB}{AC} = \frac{AD}{AC'} = \frac{a/2 - B'D}{1/2} = a - BD.$$

Using the fact that the length of the altitude CC' is $\sqrt{a^2 - \frac{1}{4}}$, this leads to

$$a^2 - 1 = aBD = 2|ABC| = \sqrt{a^2 - \frac{1}{4}},$$

or, equivalently,

$$a^2 - 1 = \sqrt{a^2 - \frac{1}{4}}.$$

Clearly, $a > 1$, and the only solution is $a = \sqrt{\frac{5}{2}}$. □

Problem 14.
In the non-isosceles triangle ABC the altitude from A meets side BC in D. Let M be the midpoint of BC and let N be the reflection of M in D. The circumcircle of the triangle AMN intersects the side AB in $P \neq A$ and the side AC in $Q \neq A$. Prove that AN, BQ and CP are concurrent.

Solution 1. Without loss of generality, we assume the order of the points on BC to be B, M, D, N, C. This implies that P is on the segment AB and Q is on the segment AC.

Since D is the midpoint of MN and AD is perpendicular to MN, the line AD is the perpendicular bisector of MN, which contains the circumcentre of $\triangle AMN$. As A is on the perpendicular bisector of MN, we have $|AM| = |AN|$. We now have $\angle APM = \angle AQN$. Therefore

$$\angle CQN = 180^\circ - \angle AQN = 180^\circ - \angle APM = \angle BPM.$$

Furthermore, as $NMPQ$ is cyclic, we have

$$\angle NQP = 180^\circ - \angle NMP = \angle BMP.$$

Hence

$$\angle AQP = 180^\circ - \angle CQN - \angle NQP = 180^\circ - \angle BPM - \angle BMP = \angle PBM = \angle ABC.$$

Similarly,

$$\angle APQ = \angle BCA.$$

Now we have $\triangle APQ \sim \triangle ACB$. So

$$\frac{|AP|}{|AQ|} = \frac{|AC|}{|AB|}.$$

Furthermore, $\angle MAB = \angle MAP = \angle MNP = \angle BNP$, so $\triangle BMA \sim \triangle BPN$, and hence

$$\frac{|BN|}{|BP|} = \frac{|BA|}{|BM|}.$$

We also have $\angle CAM = \angle QAM = 180^\circ - \angle QNM = \angle QNC$. This implies $\triangle CMA \sim \triangle CQN$, so

$$\frac{|CQ|}{|CN|} = \frac{|CM|}{|CA|}.$$

Putting everything together, we find

$$\frac{|BN|}{|BP|} \cdot \frac{|CQ|}{|CN|} \cdot \frac{|AP|}{|AQ|} = \frac{|BA|}{|BM|} \cdot \frac{|CM|}{|CA|} \cdot \frac{|AC|}{|AB|}.$$

As $|BM| = |CM|$, the right-hand side is equal to 1. This means that

$$\frac{|BN|}{|NC|} \cdot \frac{|CQ|}{|QA|} \cdot \frac{|AP|}{|PB|} = 1.$$

With Ceva's theorem we can conclude that AN, BQ and CP are concurrent. □

Solution 2. We consider the same configuration as in Solution 1. Let K be the second intersection of AD with the circumcircle of $\triangle AMN$. Since D is the midpoint of MN and AD is perpendicular to MN, the line AD is the perpendicular bisector of MN, which contains the circumcentre of $\triangle AMN$. So AK is a diameter of this circumcircle. Now we have $\angle BPK = 90^\circ = \angle BDK$, so $BPDK$ is a cyclic quadrilateral. Also, A, M, N, K, P and Q are concyclic. Using both circles, we find

$$180^\circ - \angle CQP = \angle AQP = \angle AKP = \angle DKP = \angle DBP = \angle CBP.$$

This implies that $BPQC$ is cyclic as well. Using the power theorem we find $AP \cdot AB = AQ \cdot AC$, with directed lengths. Also, $BN \cdot BM = BP \cdot BA$ and $CN \cdot CM = CQ \cdot CA$. Hence

$$AP \cdot AB \cdot BN \cdot BM \cdot CQ \cdot CA = AQ \cdot AC \cdot BP \cdot BA \cdot CN \cdot CM.$$

Changing the signs of all six lengths on the right-hand side and replacing MC by BM, we find

$$AP \cdot AB \cdot BN \cdot BM \cdot CQ \cdot CA = QA \cdot CA \cdot PB \cdot AB \cdot NC \cdot BM.$$

Cleaning this up, we have

$$AP \cdot BN \cdot CQ = QA \cdot PB \cdot NC,$$

implying

$$\frac{BN}{NC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = 1.$$

With Ceva's theorem we can conclude that AN, BQ and CP are concurrent. □

Problem 15.
In triangle ABC, the interior and exterior angle bisectors of $\angle BAC$ intersect the line BC in D and E, respectively. Let F be the second point of intersection of the line AD with the circumcircle of the triangle ABC. Let O be the circumcentre of the triangle ABC and let D' be the reflection of D in O. Prove that $\angle D'FE = 90°$.

Solution 1. Note that $AB \neq AC$, since otherwise the exterior angle bisector of $\angle BAC$ would be parallel to BC. So assume without loss of generality that $AB < AC$. Let M be the midpoint of BC and let F' the reflection of F in O, which is also the second intersection of the line AE with the circumcircle of $\triangle ABC$. We now have

$$\angle D'FO = \angle OF'D.$$

Since $\angle DMF' = 90° = \angle DAF'$, the quadrilateral $MDAF'$ is cyclic, thus

$$\angle OF'D = \angle MF'D = \angle MAD.$$

Furthermore, $\angle FME = 90° = \angle FAE$, so $FMAE$ is cyclic as well. This implies that

$$\angle MAD = \angle MAF = \angle MEF.$$

Combining these three equalities we find that $\angle D'FO = \angle MEF$, thus

$$\angle D'FE = \angle D'FO + \angle OFE = \angle MEF + \angle MFE = 180° - \angle EMF = 90°. \quad \square$$

Solution 2. Again, assume $AB < AC$ and define F' as in the previous solution. Let G be the intersection of the lines DF' and EF.

We can easily see that FA is perpendicular to EF', and BC to FF'. Now, in triangle $\triangle EFF'$, we have that FD and ED are altitudes, so D is the orthocentre of this triangle. Now, $F'D$ is an altitude as well and we find that $F'G$ is perpendicular to EF. Since FF' is a diameter of the circumcircle of $\triangle ABC$, G must lie on this circle as well.

We now find

$$\angle EFA = \angle GFA = \angle GF'A = \angle DF'A.$$

Also, $\angle DF'F = \angle D'FF'$. This implies that

$$\angle D'FE = \angle D'FF' + \angle F'FA + \angle AFE = \angle DF'F + \angle F'FA + \angle AF'D = \angle F'FA + \angle AF'F.$$

Now, in triangle $\triangle AFF'$ we have

$$\angle F'FA + \angle AF'F = 180^\circ - \angle FAF' = 90^\circ,$$

as required. $\square$

Solution 3. Again, assume $AB < AC$. We first consider the case $\angle BAC = 90^\circ$. Define F' as in the previous solution. Now O and D' lie on BC, so $\triangle D'FO$ and $\triangle DFO$ are mirror images with respect to FF', while $\triangle OFE$ and $\triangle OF'E$ are mirror images with respect to BC. We find that

$$\angle D'FE = \angle D'FO + \angle OFE = \angle DFO + \angle OF'E = \angle AFF' + \angle FF'A = 180^\circ - \angle FAF' = 90^\circ.$$

Now assume that $\angle BAC \neq 90^\circ$. We consider the configuration where $\angle BAC < 90^\circ$. Let M, N and L be the midpoints of the line segments BC, DE and DF, respectively. Note that N is the circumcentre of $\triangle ADE$, so we find

$$\begin{aligned}\angle NAF = \angle NAD = \angle NDA = \angle DAC + \angle ACD \\ = \frac{1}{2}\angle A + \angle ACD = \angle BAF + \angle ACD = \angle BCF + \angle ACD = \angle ACF.\end{aligned}$$

Hence NA is tangent to the circumcircle of $\triangle ABC$, thus $NA \perp OA$. Furthermore, we have $NM \perp OM$, so $AOMN$ is cyclic with ON as diameter. Now, since L is the circumcentre of $\triangle DMF$, we find

$$\angle LMN = \angle LMD = \angle LDM = \angle ADN = \angle DAN = \angle LAN,$$

so $ANLM$ is cyclic. Combining this with what we found before, we now conclude that $AOMLN$ is cyclic with ON as diameter, thus $\angle OLN = 90^\circ$. Using a dilation with centre D and factor 2 we now can conclude $\angle D'FE = \angle OLN = 90^\circ$.

In case $\angle BAC > 90^\circ$, the proof is similar (the cyclic quadrilateral will this time be $AMON$). $\square$

Solution 4. We consider the configuration where C, D, B and E are on the line BC in that order. The other configuration can be solved analogously. Let P and R be the feet of the perpendiculars from D' and O to the line AD, respectively, and let Q and S be the feet of the perpendiculars from D' and O to the line CD, respectively. Since D' is the reflection of D with respect to O, we have $PR = RD$. Since we also have $OA = OF$ and therefore $RA = RF$, we obtain $AD = PF$. Similarly, $CD = BQ$. By Pythagoras's theorem,

$$D'F^2 = D'P^2 + PF^2 = 4OR^2 + AD^2 = 4OA^2 - 4AR^2 + AD^2 = 4OA^2 - AF^2 + AD^2$$

and

$$D'E^2 = D'Q^2 + EQ^2 = 4OS^2 + EQ^2 = 4OB^2 - 4BS^2 + EQ^2 = 4OB^2 - BC^2 + (EB + CD)^2.$$

And since $\angle EAF = 90^\circ$ we have

$$EF^2 = AE^2 + AF^2.$$

As $OA = OB$, we conclude that

$$\begin{aligned}D'F^2 + EF^2 - D'E^2 = (4OA^2 - AF^2 + AD^2) + (AE^2 + AF^2) - (4OB^2 - BC^2 + (EB + CD)^2) \\ = AE^2 + AD^2 + BC^2 - (EB + CD)^2.\end{aligned}$$

By Pythagoras's theorem again, we obtain $AE^2 + AD^2 = ED^2$, and hence

$$D'F^2 + EF^2 - D'E^2 = ED^2 + BC^2 - (EB + CD)^2.$$

We have

$$\begin{aligned}ED^2 + BC^2 &= (EB + BD)^2 + (BD + CD)^2 \\ &= EB^2 + BD^2 + 2 \cdot EB \cdot BD + BD^2 + CD^2 + 2 \cdot BD \cdot CD \\ &= EB^2 + CD^2 + 2 \cdot BD \cdot (EB + BD + CD) \\ &= EB^2 + CD^2 + 2 \cdot BD \cdot EC.\end{aligned}$$

By the internal and external bisector theorems, we have

$$\frac{BD}{CD} = \frac{BA}{CA} = \frac{BE}{CE},$$

hence

$$ED^2 + BC^2 = EB^2 + CD^2 + 2 \cdot CD \cdot BE = (EB + CD)^2.$$

So

$$D'F^2 + EF^2 - D'E^2 = 0,$$

which implies that $\angle D'FE = 90°$. □

Problem 16.
Denote by $P(n)$ the greatest prime divisor of n. Find all integers $n \geq 2$ for which

$$P(n) + \lfloor \sqrt{n} \rfloor = P(n+1) + \lfloor \sqrt{n+1} \rfloor.$$

(Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

Solution. Answer: The equality holds only for $n = 3$.

It is easy to see that $P(n) \neq P(n+1)$. Therefore we need also that $\lfloor \sqrt{n} \rfloor \neq \lfloor \sqrt{n+1} \rfloor$ in order for equality to hold. This is only possible if $n+1$ is a perfect square. In this case,

$$\lfloor \sqrt{n} \rfloor + 1 = \lfloor \sqrt{n+1} \rfloor,$$

and hence $P(n) = P(n+1) + 1$. As both $P(n)$ and $P(n+1)$ are primes, it must be that $P(n) = 3$ and $P(n+1) = 2$.

It follows that $n = 3^a$ and $n + 1 = 2^b$, and we are required to solve the equation $3^a = 2^b - 1$. Calculating modulo 3, we find that b is even. Put $b = 2c$:

$$3^a = (2^c - 1)(2^c + 1).$$

As both factors cannot be divisible by 3 (their difference is 2), $2^c - 1 = 1$. From this we get $c = 1$, which leads to $n = 3$. □

Problem 17.
Find all positive integers n for which $n^{n-1} - 1$ is divisible by 2^{2015}, but not by 2^{2016}.

Solution. Since n must be odd, write $n = 2^d u + 1$, where $u, d \in \mathbf{N}$ and u is odd. Now

$$n^{n-1} - 1 = (n^{2^d} - 1)(\underbrace{n^{2^d \cdot (u-1)} + \cdots + n^{2^d \cdot 1} + 1}_{u}),$$

and hence $2^{2015} \parallel (n^{n-1} - 1)$ iff $2^{2015} \parallel (n^{2^d} - 1)$. (The notation $p^k \parallel m$ denotes that $p^k \mid m$ and $p^{k+1} \nmid m$.)

We factorise once more:

$$\begin{aligned} n^{2^d} - 1 &= (n-1)(n+1)\underbrace{(n^2+1)\cdots(n^{2^{d-1}}+1)}_{d-1} \\ &= 2^d u \cdot 2(2^{d-1}u+1)\underbrace{(n^2+1)\cdots(n^{2^{d-1}}+1)}_{d-1}. \end{aligned}$$

If $k \geq 1$, then $2 \parallel n^{2^k} + 1$, and so from the above

$$2^{2d} \parallel 2^d u \cdot 2 \cdot \underbrace{(n^2+1)\cdots(n^{2^{d-1}}+1)}_{d-1} \qquad \text{and} \qquad 2^{2015-2d} \parallel (2^{d-1}u+1).$$

It is easy to see that this is the case exactly when $d = 1$ and $u = 2^{2013}v - 1$, where v is odd.

Hence the required numbers are those of the form

$$n = 2(2^{2013}v - 1) + 1 = 2^{2014}v - 1,$$

for v a positive odd number. □

Problem 18.
Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 1$ with n (not necessarily distinct) integer roots. Assume that there exist distinct primes $p_0, p_1, \ldots, p_{n-1}$ such that $a_i > 1$ is a power of p_i, for all $i = 0, \ldots, n-1$. Find all possible values of n.

Solution. Obviously all the roots have to be negative by the positivity of the coefficients. If at least two of the roots are unequal to -1, then both of them have to be powers of p_0. Now Vieta's formulæ yield $p_0 \mid a_1$, which is a contradiction. Thus we can factor f as

$$f(x) = (x + a_0)(x+1)^{n-1}.$$

Expanding yields

$$a_2 = \binom{n-1}{1} + a_0\binom{n-1}{2} \qquad \text{and} \qquad a_{n-2} = a_0\binom{n-1}{n-2} + \binom{n-1}{n-3}.$$

If $n \geq 5$, we see that $2 \neq n-2$ and so the two coefficients above are relatively prime, being powers of two distinct primes. However, depending on the parity of n, we have that a_2 and a_{n-2} are both divisible by $n-1$ or $\frac{n-1}{2}$, which is a contradiction.

For $n = 1, 2, 3, 4$, the following polynomials meet the requirements:

$$\begin{aligned} f_1(x) &= x + 2 \\ f_2(x) &= (x+2)(x+1) = x^2 + 3x + 2 \\ f_3(x) &= (x+3)(x+1)^2 = x^3 + 5x^2 + 7x + 3 \\ f_4(x) &= (x+2)(x+1)^3 = x^4 + 5x^3 + 9x^2 + 7x + 2 \end{aligned}$$

□

Problem 19.
Three pairwise distinct positive integers a, b, c, with $\gcd(a, b, c) = 1$, satisfy

$$a \mid (b-c)^2, \qquad b \mid (c-a)^2 \qquad \text{and} \qquad c \mid (a-b)^2.$$

Prove that there does not exist a non-degenerate triangle with side lengths a, b, c.

Solution. First observe that these numbers are pairwise coprime. Indeed, if, say, a and b are divisible by a prime p, then p divides b, which divides $(a-c)^2$; hence p divides $a-c$, and therefore p divides c. Thus, p is a common divisor of these three numbers, a contradiction.

Now consider the number

$$M = 2ab + 2bc + 2ac - a^2 - b^2 - c^2.$$

It is clear from the problem condition that M is divisible by a, b, c, and therefore M is divisible by abc.

Assume that a triangle with sides a, b, c exists. Then $a < b + c$, and so $a^2 < ab + ac$. Analogously, we have $b^2 < bc + ba$ and $c^2 < ca + cb$. Summing these three inequalities leads to $M > 0$, and hence $M \geq abc$.

On the other hand,

$$a^2 + b^2 + c^2 > ab + bc + ac,$$

and therefore $M < ab + bc + ac$. Supposing, with no loss of generality, $a > b > c$, we must have $M < 3ab$. Taking into account the inequality $M \geq abc$, we conclude that $c = 1$ or $c = 2$ are the only possibilities.

For $c = 1$ we have $b < a < b + 1$ (the first inequality is our assumption, the second is the triangle inequality), a contradiction.

For $c = 2$ we have $b < a < b + 2$, i.e. $a = b + 1$. But then $1 = (a-b)^2$ is not divisible by $c = 2$. □

Problem 20.

For any integer $n \geq 2$, we define A_n to be the number of positive integers m with the following property: the distance from n to the nearest non-negative multiple of m is equal to the distance from n^3 to the nearest non-negative multiple of m. Find all integers $n \geq 2$ for which A_n is odd.

(Note: The distance between two integers a and b is defined as $|a - b|$.)

Solution. For an integer m we consider the distance d from n to the nearest multiple of m. Then $m \mid n \pm d$, which means $n \equiv \pm d \bmod m$. So if, for some m, the distance from n to the nearest multiple of m is equal to the distance from n^3 to the nearest multiple of m, then $n \equiv \pm n^3 \bmod m$.

On the other hand, if $n \equiv \pm n^3 \bmod m$, then there exists a $0 \leq d \leq \frac{1}{2}m$ such that $n \equiv \pm d \bmod m$ and $n^3 \equiv \pm d \bmod m$, so the distance from n to the nearest multiple of m is equal to the distance from n^3 to the nearest multiple of m.

We conclude that we need to count the number of positive integers m such that $n \equiv \pm n^3 \bmod m$, or, equivalently, $m \mid n^3 - n$ or $m \mid n^3 + n$. That is,

$$\begin{aligned} A_n &= \left|\left\{\, m \in \mathbf{Z}^+ \;\middle|\; m \mid n^3 - n \text{ or } m \mid n^3 + n \,\right\}\right| \\ &= \left|\left\{\, m \in \mathbf{Z}^+ \;\middle|\; m \mid n^3 - n \,\right\}\right| + \left|\left\{\, m \in \mathbf{Z}^+ \;\middle|\; m \mid n^3 + n \,\right\}\right| - \left|\left\{\, m \in \mathbf{Z}^+ \;\middle|\; m \mid n^3 - n \text{ and } m \mid n^3 + n \,\right\}\right| \\ &= \left|\left\{\, m \in \mathbf{Z}^+ \;\middle|\; m \mid n^3 - n \,\right\}\right| + \left|\left\{\, m \in \mathbf{Z}^+ \;\middle|\; m \mid n^3 + n \,\right\}\right| - \left|\left\{\, m \in \mathbf{Z}^+ \;\middle|\; m \mid \gcd(n^3 - n, n^3 + n) \,\right\}\right| \\ &= \tau(n^3 - n) + \tau(n^3 + n) - \tau(\gcd(n^3 - n, n^3 + n)), \end{aligned}$$

where $\tau(k)$ denotes the number of (positive) divisors of a positive integer k.

Recall that $\tau(k)$ is odd if and only if k is a square. Furthermore, we have

$$\gcd(n, n^2 \pm 1) = 1.$$

So if $n^3 \pm n$ were a square, then both n and $n^2 \pm 1$ would be squares. But $n^2 \pm 1$ is not a square, since $n \geq 2$ and the only consecutive squares are $0, 1$. Hence neither $n^3 - n$ nor $n^3 + n$ is a square, so the first two terms $\tau(n^3 - n)$ and $\tau(n^3 + n)$ are both even. Hence A_n is odd if and only if $\gcd(n^3 - n, n^3 + n)$ is a square.

We have

$$\gcd(n^2-1, n^2+1) = \gcd(n^2-1, 2) = \begin{cases} 1 & \text{if } n \text{ even,} \\ 2 & \text{if } n \text{ odd.} \end{cases}$$

Hence,

$$\gcd(n^3-n, n^3+n) = \begin{cases} n & \text{if } n \text{ even,} \\ 2n & \text{if } n \text{ odd.} \end{cases}$$

Note that $2n$ for n odd is never a square, since it has exactly one factor of 2. We conclude that A_n is odd if and only if n is an even square. □