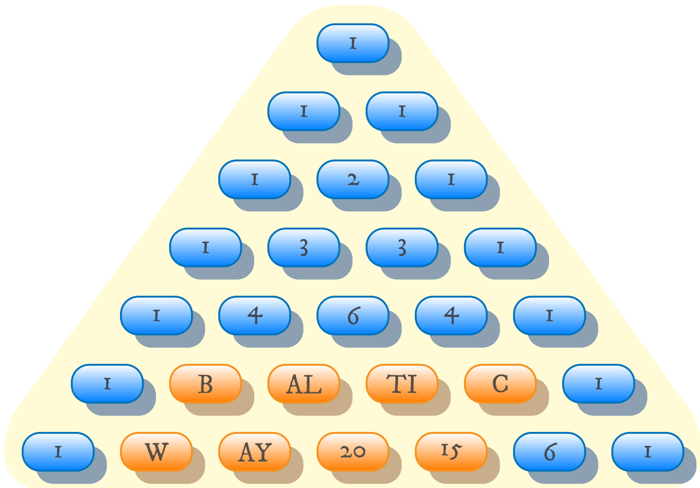

Baltic Way

1990–2014

Jubilee Selection

Last year marked the completion of the 25th Baltic Way. These have been twenty-five wonderful years of beautiful, challenging problems seeking their solution. Young mathematicians from all across Northern Europe have been trying their combined wits against those of their neighbouring countries, for Baltic Way is unique among mathematics competitions in being a true team contest.

These twenty-five years have, of course, also seen an array of festive banquets and joyous activities in between the hard mathematics. (The *Mafia* game is, we have observed, a frequent contender.) Many young mathematicians took their first budding steps towards a professional career at these contests, and friendships formed during those eventful days will often, no doubt, last a lifetime.

As a venerable way to celebrate twenty-five years of past Baltic Ways, the Organisation Committee have decided this year to host a special event, called the *Baltic Way Beauty Contest*. We invite leaders, deputy leaders and contestants alike to vote for the most beautiful problems among the 499 that have occurred in the contest ever since its inception in 1990.

Why only 499, the arithmetically inclined reader may ask, why not a full $500 = 20 \cdot 25$? Because the twentieth problem of the very first Baltic Way, back in 1990, was exceptionally open-ended:

> A creative task: propose an original competition problem together with its solution.

We have requested the participating countries to nominate their favourite problems. They were asked to select such as

- are simply beautiful;
- have quick and elegant solutions;
- excel in originality and innovation;
- or whose statement contains an element of great surprise.

The nominated problems will be found on the subsequent pages, divided into the traditional categories: Algebra, Combinatorics, Geometry and Number Theory. We ask all the participants of Baltic Way 2015 to cast their votes.

And may we all hope that Baltic Way will continue to live for another twenty-five years (at the very least).

Algebra

Problem A.1 *Baltic Way 1990*

The problem looks daunting at first, until one realises that the recurrence formula bears a haunting familiarity to the angle addition formula for the tangent function. Then the solution is a one-liner.

Problem. Let $a_0 > 0$, $c > 0$ and

$$a_{n+1} = \frac{a_n + c}{1 - a_n c}, \qquad n = 0, 1, \ldots.$$

Is it possible that the first 1990 terms a_0, a_1, …, a_{1989} are all positive, but $a_{1990} < 0$?

Solution. Obviously we can find angles $0 < \alpha, \beta < 90^\circ$ such that

$$\tan\alpha > 0, \qquad \tan(\alpha + \beta) > 0, \qquad \ldots, \qquad \tan(\alpha + 1989\beta) > 0,$$

but $\tan(\alpha + 1990\beta) < 0$. Now it suffices to note that if we take $a_0 = \tan\alpha$ and $c = \tan\beta$, then $a_n = \tan(\alpha + n\beta)$. □

Problem A.2 *Baltic Way 1990*

Another problem with an elegant one-line solution, deploying a smattering of calculus.

Problem. Prove that, for any real $a_1, a_2, \ldots, a_n$,

$$\sum_{i,j=1}^{n} \frac{a_i a_j}{i + j - 1} \geqslant 0.$$

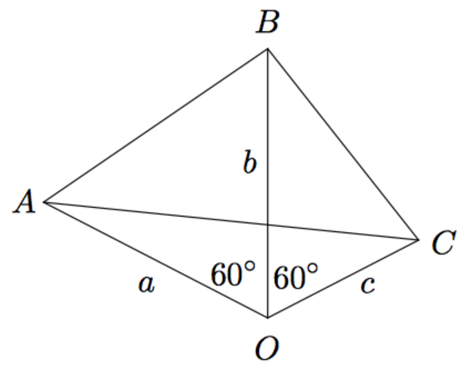

FIGURE 1: Problem A.3.

Solution. Consider the polynomial

$$P(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}.$$

Then

$$P^2(x) = \sum_{i,j=1}^{n} a_i a_j x^{i+j-2}$$

and

$$\int_0^1 P^2(x)\,dx = \sum_{i,j=1}^{n} \frac{a_i a_j}{i+j-1}. \quad \square$$

Problem A.3 *Baltic Way 2000*

The arsenal of known inequalities seems to no avail. Squaring the inequality leads to a formidable mess. The solution is a marvel.

Problem. Prove that for all positive real numbers a, b, c, we have

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geqslant \sqrt{a^2 + ac + c^2}.$$

Solution. If $|OA| = a$, $|OB| = b$, $|OC| = c$ (see Figure 1), then the inequality follows from $|AC| \leqslant |AB| + |BC|$ by applying the Law of Cosines to triangles AOB, BOC and AOC. The same argument holds if the quadrangle $OABC$ is concave. $\square$

Problem A.4 *Baltic Way 2008*

Baltic Way has its own take on the love story of Romeo and Juliet. Great care was exercised when phrasing the problem, so that its romantic flavour would not be lost.

Surprisingly, Romeo's and Juliet's numbers must match. This would be false if negative numbers were permitted, since then Juliet's numbers might be the negative of Romeo's, and they would still have identical tetrahedra.

Problem. Suppose that Romeo and Juliet each have a regular tetrahedron, to the vertices of which some positive real numbers are assigned. They associate, to each edge of their tetrahedra, the product of the two numbers assigned to its end points. Then they write on each face of their tetrahedra the sum of the three numbers associated to its three edges. The four numbers written on the faces of Romeo's tetrahedron turn out to coincide with the four numbers written on Juliet's tetrahedron. Does it follow that the four numbers assigned to the vertices of Romeo's tetrahedron are identical to the four numbers assigned to the vertices of Juliet's tetrahedron?

Solution. Answer: yes.

Let us prove that this conclusion can in fact be drawn. For this purpose we denote the numbers assigned to the vertices of Romeo's tetrahedron by r_1, r_2, r_3, r_4 and the numbers assigned to the vertices of Juliet's tetrahedron by j_1, j_2, j_3, j_4 in such a way that

$$r_2r_3 + r_3r_4 + r_4r_2 = j_2j_3 + j_3j_4 + j_4j_2 \quad (1)$$
$$r_1r_3 + r_3r_4 + r_4r_1 = j_1j_3 + j_3j_4 + j_4j_1 \quad (2)$$
$$r_1r_2 + r_2r_4 + r_4r_1 = j_1j_2 + j_2j_4 + j_4j_1 \quad (3)$$
$$r_1r_2 + r_2r_3 + r_3r_1 = j_1j_2 + j_2j_3 + j_3j_1. \quad (4)$$

We intend to show that $r_1 = j_1$, $r_2 = j_2$, $r_3 = j_3$ and $r_4 = j_4$, which clearly suffices to establish our claim. Now let

$$R = \{\, i \mid r_i > j_i \,\}$$

denote the set of indices where Romeo's corresponding number is larger and define similarly

$$J = \{\, i \mid r_i < j_i \,\}.$$

If we had $|R| > 2$, then w.l.o.g. $\{1,2,3\} \subseteq R$, which easily contradicts (4). Therefore $|R| \leqslant 2$, so let us suppose for the moment that $|R| = 2$. Then w.l.o.g. $R = \{1,2\}$, i.e. $r_1 > j_1$, $r_2 > j_2$, $r_3 \leqslant j_3$, $r_4 \leqslant j_4$. It follows that $r_1 r_2 - r_3 r_4 > j_1 j_2 - j_3 j_4$, but (1) + (2) + (3) − (4) actually tells us that both sides of this strict inequality are equal.

This contradiction yields $|R| \leqslant 1$ and, replacing the roles Romeo and Juliet played in the argument just performed, we similarly infer $|J| \leqslant 1$. For these reasons, at least two of the four desired equalities hold, say $r_1 = j_1$ and $r_2 = j_2$. Now, using (3) and (4), we easily get $r_3 = j_3$ and $r_4 = j_4$ as well. □

Problem A.5 *Baltic Way 1998*

This is one of the most surprising algebra problems, expressing an astonishing geometrical fact.

Problem. Let $n \geqslant 4$ be an even integer. A regular n-gon and a regular $(n-1)$-gon are inscribed into the unit circle. For each vertex of the n-gon, consider the distance from this vertex to the nearest vertex of the $(n-1)$-gon, measured along the circumference. Let S be the sum of these n distances. Prove that S depends only on n, and not on the relative position of the two polygons.

Solution. Identify each point of the n-gon $P_1 \dots P_n$ and of the $(n-1)$-gon $Q_1 \dots Q_{n-1}$ (in anti-clockwise order) with its angle, measured relative the point Q_1. Let $\theta = \frac{2\pi}{n(n-1)}$, and put $P_{\frac{n}{2}+1} = \alpha$. Calculating modulo $n\theta = \frac{2\pi}{n-1}$ will identify all the points Q_k with 0, while the points

$$P_k = \alpha + \left(k - \frac{n}{2}\right)\frac{2\pi}{n} = \alpha + \left(k - \frac{n}{2}\right)(n-1)\theta \equiv \alpha - \left(k - \frac{n}{2}\right)\theta, \quad k = 1, \dots, n,$$

will be equally spaced out in the interval $[0, n\theta)$. Exactly half of them will be closest to the left end-point 0 and half of them will be closest to the right end-point $n\theta$. Rotating the n-gon will increase the former distances and decrease the latter by the same amount. Hence the sum will remain constant.

Explicitly, we may suppose, with no loss of generality, that $0 \leqslant \alpha < \theta$. Then

$$0 \leqslant P_{\frac{n}{2}} = \alpha < P_{\frac{n}{2}-1} = \alpha + \theta < \cdots < P_1 = \alpha + \left(\frac{n}{2} - 1\right)\theta < \frac{1}{2}n\theta,$$

and the sum of the their distances to the closest vertex Q_j is

$$\alpha + (\alpha + \theta) + \cdots + \left(\alpha + \left(\frac{n}{2} - 1\right)\theta\right) = \frac{1}{2}n\alpha + \left(\frac{1}{8}n^2 - \frac{1}{4}n\right)\theta.$$

Also,

$$-\frac{1}{2}n\theta \leqslant P_n = \alpha - \frac{n}{2}\theta < \cdots < P_{\frac{n}{2}+2} = \alpha - 2\theta < P_{\frac{n}{2}+1} = \alpha - \theta < 0,$$

and the sum of the distances to the closest vertex Q_j is

$$(\theta - \alpha) + (2\theta - \alpha) + \cdots + \left(\frac{n}{2}\theta - \alpha\right) = -\frac{1}{2}n\alpha + \left(\frac{1}{8}n^2 + \frac{1}{4}n\right)\theta.$$

The sum sought for is

$$S = \frac{1}{2}n\alpha + \left(\frac{1}{8}n^2 - \frac{1}{4}n\right)\theta - \frac{1}{2}n\alpha + \left(\frac{1}{8}n^2 + \frac{1}{4}n\right)\theta = \frac{1}{4}n^2\theta = \frac{n\pi}{2(n-1)}. \quad \square$$

Combinatorics

Problem C.1 *Baltic Way 1999*

Here we have combinatorial geometry at its best. The problem and its solution are crisp and succinct.

Problem. Can the points of a disc of radius 1 (including its circumference) be partitioned into three subsets in such a way that no subset contains two points separated by distance 1?

Solution. Answer: no.

Let O denote the centre of the disc, and $P_1, \ldots, P_6$ the vertices of an inscribed regular hexagon in the natural order (see Figure 2).

If the required partitioning exists, then $\{O\}$, $\{P_1, P_3, P_5\}$ and $\{P_2, P_4, P_6\}$ are contained in different subsets. Now consider the circles of radius 1 centred in P_1, P_3 and P_5. The circle of radius $1/\sqrt{3}$ centred in O intersects these three

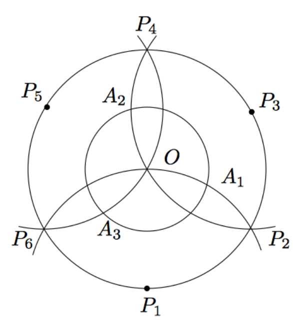

FIGURE 2: Problem C.1.

circles in the vertices A_1, A_2, A_3 of an equilateral triangle of side length 1. The vertices of this triangle belong to different subsets, but none of them can belong to the same subset as P_1 — a contradiction. Hence the required partitioning does not exist. □

Problem C.2 *Baltic Way 1994*

The infamous spy problem of 1994 is astounding indeed. If any ten spies, out of sixteen, spy cyclically on each other, then so do any eleven spies. The figures are judiciously chosen so as to make the problem work.

Problem. The Wonder Island Intelligence Service has 16 spies in Tartu. Each of them watches on some of his colleagues. It is known that if spy A watches on spy B then B does not watch on A. Moreover, any 10 spies can be numbered in such a way that the first spy watches on the second, the second watches on the third, ..., the tenth watches on the first. Prove that any 11 spies can also be numbered in a similar manner.

Solution. We call two spies A and B *neutral* to each other if neither A watches on B nor B watches on A.

Denote the spies $A_1, A_2, \ldots, A_{16}$. Let a_i, b_i and c_i denote the number of spies that watch on A_i, the number of spies that are watched by A_i and the number of spies neutral to A_i, respectively. Clearly, we have

$$\begin{aligned} a_i + b_i + c_i &= 15, \\ a_i + c_i &\leqslant 8, \\ b_i + c_i &\leqslant 8 \end{aligned}$$

for any $i = 1, \ldots, 16$ (if any of the last two inequalities does not hold then there exist 10 spies who cannot be numbered in the required manner). Combining the relations above we find $c_i \leqslant 1$. Hence, for any spy, the number of his neutral colleagues is 0 or 1.

Now suppose there is a group of 11 spies that cannot be numbered as required. Let B be an arbitrary spy in this group. Number the other 10 spies as $C_1, C_2, \ldots, C_{10}$ so that C_1 watches on $C_2, \ldots, C_{10}$ watches on C_1. Suppose there is no spy neutral to B among $C_1, \ldots, C_{10}$. Then, if C_1 watches on B then B cannot watch on C_2, as otherwise $C_1, B, C_2, \ldots, C_{10}$ would form an 11-cycle.

So C_2 watches on B, etc. As some of the spies $C_1, C_2, \ldots, C_{10}$ must watch on B we get all of them watching on B, a contradiction. Therefore, each of the 11 spies must have exactly one spy neutral to him among the other 10 — but this is impossible. □

Problem C.3 *Baltic Way 2003*

The beauty of this problem lies in two facts. First, that the requested number n can be identified at all under the strained conditions. Second, that the ubiquitous Fibonacci numbers make a remarkable, and very surprising, appearance.

Problem. It is known that n is a positive integer, $n \leqslant 144$. Ten questions of type "Is n smaller than a?" are allowed. Answers are given with a delay: The answer to the i'th question is given only after the $(i+1)$'st question is asked, $i = 1, 2, \ldots, 9$. The answer to the tenth question is given immediately after it is asked. Find a strategy for identifying n.

Solution. Let the Fibonacci numbers be denoted $F_0 = 1$, $F_1 = 2$, $F_2 = 3$ etc. Then $F_{10} = 144$. We will prove by induction on k that using k questions subject to the conditions of the problem, it is possible to determine any positive integer $n \leqslant F_k$. First, for $k = 0$ it is trivial, since without asking we know that $n = 1$. For $k = 1$, we simply ask if n is smaller than 2. For $k = 2$, we ask if n is smaller than 3 and if n is smaller than 2; from the two answers we can determine n.

Now, in general, our first two questions will always be "Is n smaller than $F_{k-1} + 1$?" and "Is n smaller than $F_{k-2} + 1$?". We then receive the answer to the first question. As long as we receive affirmative answers to the $(i-1)$'st question, the $(i+1)$'st question will be "Is n smaller than $F_{k-(i+1)} + 1$?". If at any point, say after asking the j'th question, we receive a negative answer to the $(j-1)$'st question, we then know that

$$F_{k-(j-1)} + 1 \leqslant n \leqslant F_{k-(j-2)},$$

so n is one of

$$F_{k-(j-2)} - F_{k-(j-1)} = F_{k-j}$$

consecutive integers, and by induction we may determine n using the remaining $k-j$ questions. Otherwise, we receive affirmative answers to all the questions, the last being "Is n smaller than $F_{k-k}+1=2$?"; so $n=1$ in that case. □

Problem C.4 *Baltic Way 2010*

Problems concerning cities connected by roads or direct flights — graph theory in disguise — abound in the Baltic Way. This problem has quite an unexpected conclusion.

Problem. There are some cities in a country; one of them is the capital. For any two cities A and B there is a direct flight from A to B and a direct flight from B to A, both having the same price. Suppose that all round trips with exactly one landing in every city have the same total cost. Prove that all round trips that miss the capital and with exactly one landing in every remaining city cost the same.

Solution. Let C be the capital and C_1, C_2, $\dots$, C_n be the remaining cities. Denote by $d(x,y)$ the price of the connection between the cities x and y, and let σ be the total price of a round trip going exactly once through each city.

Now consider a round trip missing the capital and visiting every other city exactly once; let s be the total price of that trip. Suppose C_i and C_j are two consecutive cities on the route. Replacing the flight $C_i \to C_j$ by two flights: from C_i to the capital and from the capital to C_j, we get a round trip through all cities, with total price σ. It follows that

$$\sigma = s + d(C,C_i) + d(C,C_j) - d(C_i,C_j),$$

so it remains to show that the quantity

$$\alpha(i,j) = d(C,C_i) + d(C,C_j) - d(C_i,C_j)$$

is the same for all 2-element subsets $\{i,j\} \subset \{1,2,\dots,n\}$.

For this purpose, note that $\alpha(i,j)=\alpha(i,k)$ whenever i,j,k are three distinct indices; indeed, this equality is equivalent to

$$d(C_j,C) + d(C,C_i) + d(C_i,C_k) = d(C_j,C_i) + d(C_i,C) + d(C,C_k),$$

which is true by considering any trip from C_k to C_j going through all cities except C and C_i exactly once and completing this trip to a round trip in two ways:

$$C_j \to C \to C_i \to C_k \qquad \text{and} \qquad C_j \to C_i \to C \to C_k.$$

Therefore the values of α coincide on any pair of 2-element sets sharing a common element. But then clearly $\alpha(i,j) = \alpha(i,j') = \alpha(i',j')$ for all indices i, j, i', j' with $i \neq j$, $i' \neq j'$, and the solution is complete. □

Problem C.5 *Baltic Way 2002*

Now for a mathemagical trick! Two magicians — or are they mathematicians? — perform the seemingly impossible. No less than three solutions are presented to this problem: one constructive, based on arithmetic modulo 5, one non-constructive, making clever use of Hall's Matching Theorem, and finally, another one constructive, and amazingly simple at that, such as even a magician of the world might employ.

Problem. Two magicians show the following trick. The first magician goes out of the room. The second magician takes a deck of 100 cards labelled by numbers 1, 2, …, 100 and asks three spectators to choose in turn one card each. The second magician sees what card each spectator has taken. Then he adds one more card from the rest of the deck. Spectators shuffle these 4 cards, call the first magician and give him these 4 cards. The first magician looks at the 4 cards and "guesses" what card was chosen by the first spectator, what card by the second and what card by the third. Prove that the magicians can perform this trick.

Solution 1. We will identify ourselves with the second magician. Then we need to choose a card in such a manner that another magician will be able to understand which of the 4 cards we have chosen and what information it gives about the order of the other cards. We will reach these two goals independently. Let a, b, c be the remainders of the labels of the spectators' three cards modulo 5. There are three possible cases.

Case 1: All the three remainders coincide. Then choose a card with a remainder not equal to the remainder of spectators' cards. Denote this remainder d.

Note that we now have *2 different remainders*, one of them in 3 copies (this will be used by the first magician to distinguish between the three cases). To determine which of the cards is chosen by us is now a simple exercise in division by 5. But we must also encode the ordering of the spectators' cards. These cards have a natural ordering by their labels, and they are also ordered by their belonging to the spectators. Thus, we have to encode a permutation of 3 elements. There are 6 permutations of 3 elements, let us enumerate them somehow. Then, if we want to inform the first magician that the spectators form permutation number k with respect to the natural ordering, we choose card number $5k + d$.

Case 2: The remainders a, b, c are pairwise different. Then it is clear that exactly one of the following possibilities takes place: either

$$|b-a| = |a-c|, \qquad |a-b| = |b-c|, \qquad \text{or} \qquad |a-c| = |c-b| \tag{5}$$

(the equalities are considered modulo 5). It is not hard to prove it by a case study, but one could also imagine choosing three vertices of a regular pentagon — these vertices always form an isosceles, but not an equilateral triangle.

Each of these possibilities has one of the remainders distinguished from the other two remainders (these distinguished remainders are a, b, c, respectively). Now, choose a card from the rest of the deck having the distinguished remainder modulo 5. Hence, we have three different remainders, one of them distinguished by (5) and presented in two copies. Let d be the distinguished remainder and $s = 5m + d$ be the spectator's card with this remainder.

Now we have to choose a card r with the remainder d such that the first magician would be able to understand which of the cards s and r was chosen by us and what permutation of spectators it implies. This can be done easily: if we want to inform the first magician that the spectators form permutation number k with respect to the natural ordering, we choose card number $s + 5k \bmod 100$.

The decoding procedure is easy: if we have two numbers p and q that have the same remainder modulo 5, calculate $p - q \bmod 100$ and $q - p \bmod 100$. If $p-q \bmod 100 > q-p \bmod 100$ then $r = q$ is our card and $s = p$ is the spectator's card. (The case $p - q \bmod 100 = q - p \bmod 100$ is impossible since the sum of these numbers is equal to 100, and one of them is not greater than $6 \cdot 5 = 30$.)

Case 3: Two remainders (say, a and b) coincide. Let us choose a card with the remainder $d = (a+c)/2 \bmod 5$. Then $|a-d| = |d-c| \bmod 5$, so the remainder d is distinguished by (5). Hence we have three different remainders, one of them distinguished by (5) and one of the non-distinguished remainders presented in two copies. The first magician will easily determine our card, and the rule to choose the card in order to enable him to also determine the order of the spectators is similar to the one in the first case. □

Solution 2. This solution gives a non-constructive proof that the trick is possible. For this, we need to show there is an injective mapping from the set of ordered triples to the set of unordered quadruples that additionally respects inclusion.

To prove that the desired mapping exists, let's consides a bipartite graph such that the set of ordered triples T and the set of unordered quadruples Q form the two disjoint sets of vertices and there is an edge between a triple and a quadruple if and only if the triple is a subset of the quadruple.

For each triple $t \in T$, we can add any of the remaining 97 cards to it, and thus we have 97 different quadruples connected to each triple in the graph. Conversely, for each quadruple $q \in Q$, we can remove any of the 4 cards from it, and reorder the remaining 3 cards in $3! = 6$ different ways, and thus we have 24 different triples connected to each quadruple in the graph.

According to Hall's theorem, a bipartite graph $G = (T, Q, E)$ has a perfect matching if and only if for each subset $T' \subseteq T$ the set of neighbours of T', denoted $N(T')$, satisfies $|N(T')| \geqslant |T'|$.

To prove that this condition holds for our graph, consider any subset $T' \subseteq T$. Because we have 97 quadruples for each triple, and there can be at most 24 copies of each of them in the multiset of neighbours, we have $|N(T')| \geqslant \frac{97}{24}|T'| > 4|T'|$, which is even much more than we need. Thus, the desired mapping is guaranteed to exist. □

Solution 3. Let the three chosen numbers be (x_1, x_2, x_3). At least one of the sets

$$\{1, 2, \ldots, 24\}, \quad \{25, 26, \ldots, 48\}, \quad \{49, 50, \ldots, 72\} \quad \text{and} \quad \{73, 74, \ldots, 96\}$$

should contain none of x_1, x_2 and x_3; let S be such set. Next we split S into 6 parts: $S = S_1 \cup S_2 \cup \cdots \cup S_6$ so that the first four elements of S are in S_1, the next four in S_2, etc. Now we choose $i \in \{1, 2, \ldots, 6\}$ corresponding to the order of numbers x_1, x_2 and x_3 (if $x_1 < x_2 < x_3$ then $i = 1$, if $x_1 < x_3 < x_2$ then $i = 2, \ldots,$ if $x_3 < x_2 < x_1$ then $i = 6$). Lastly, let j be the number of elements in $\{x_1, x_2, x_3\}$ that are greater than the elements of S (note that any x_k, $k \in \{1, 2, 3\}$, is either greater or smaller than all the elements of S). Now we choose $x_4 \in S_i$ so that $x_1 + x_2 + x_3 + x_4 \equiv j \bmod 4$ and add the card number x_4 to those three cards.

Decoding of $\{a, b, c, d\}$ is straightforward. We first put the numbers into increasing order and then calculate $a + b + c + d \bmod 4$ showing the added card. The added card belongs to some S_i ($i \in \{1, 2, \ldots, 6\}$) for some S and i shows us the initial ordering of cards. □

Geometry

Problem G.1 *Baltic Way 2011*

This problem may be solved in many ways: for instance, using some algebra or trusty old trigonometry. The beauty lies in the synthetic solution, a lovely specimen of transformation geometry.

Problem. Let P be a point inside a square $ABCD$ such that $PA : PB : PC$ is $1 : 2 : 3$. Determine the angle $\angle BPA$.

Solution 1. Rotate the triangle ABP by 90° around B such that A goes to C and P is mapped to a new point Q. Then

$$\angle PBQ = \angle PBC + \angle CBQ = \angle PBC + \angle ABP = 90^\circ.$$

Hence the triangle PBQ is an isosceles right-angled triangle, and $\angle BQP = 45^\circ$.

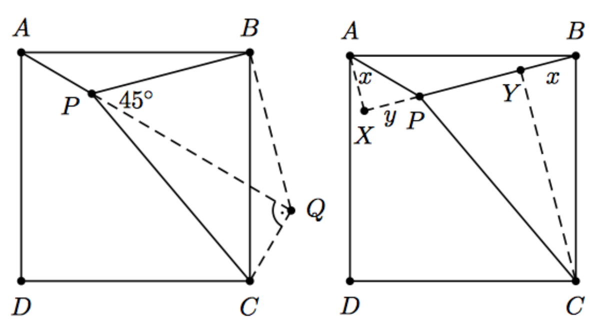

Figure 3: Problem G.1.

By Pythagoras $PQ^2 = 2PB^2 = 8AP^2$. Since

$$CQ^2 + PQ^2 = AP^2 + 8AP^2 = 9AP^2 = PC^2,$$

by the converse Pythagoras PQC is a right-angled triangle, and hence

$$\angle BPA = \angle BQC = \angle BQP + \angle PQC = 45^\circ + 90^\circ = 135^\circ. \quad \square$$

Solution 2. Let X and Y be the feet of the perpendiculars drawn from A and C to PB. Put $x = AX$ and $y = XP$. Suppose without loss of generality that $PA = 1$, $PB = 2$, and $PC = 3$. Since the right-angled triangles ABX and BCY are congruent, we have $BY = x$ and $CY = 2 + y$. Applying Pythagoras's Theorem to the triangles APX and PYC, we get $x^2 + y^2 = 1$ and $(2 - x)^2 + (2 + y)^2 = 9$. Substituting the former equation into the latter, we infer $x = y$, which in turn discloses $\angle BPA = 135^\circ$. $\square$

Problem G.2 *Baltic Way 1999*

Some problems are surprising because they yield strong and unexpected conclusions. This problem, by contrast, is surprising because it yields such a *trivial* conclusion — almost obvious, it would seem.

Problem. Prove that for any four points in the plane, no three of which are collinear, there exists a circle such that three of the four points are on the circumference and the fourth point is either on the circumference or inside the circle.

Solution 1. Consider a circle containing all these four points in its interior. First, decrease its radius until at least one of these points (say, A) will be on the circle. If the other three points are still in the interior of the circle, then rotate the circle around A (with its radius unchanged) until at least one of the other three points (say, B) will also be on the circle. The centre of the circle now lies on the perpendicular bisector of the segment AB — moving the centre along that perpendicular bisector (and changing its radius at the same time, so that points A and B remain on the circle) we arrive at a situation where at least one of the remaining two points will also be on the circle (see Figure 4). $\square$

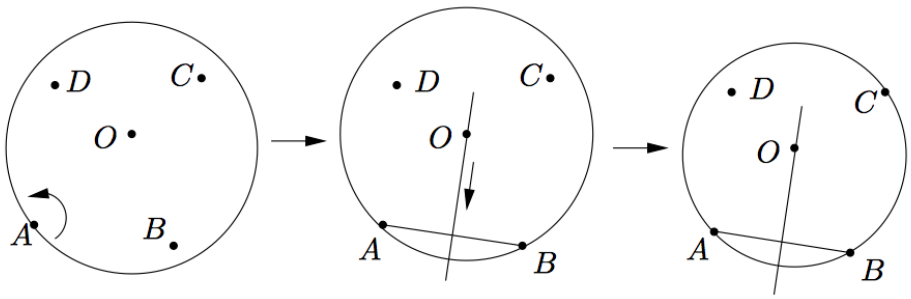

Figure 4: Problem G.2.

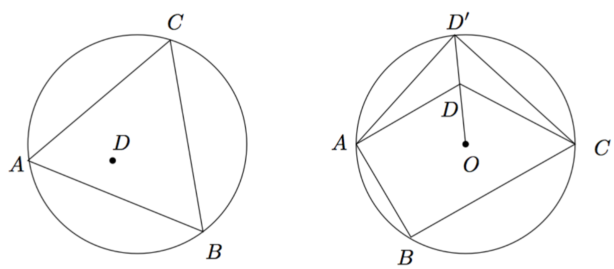

Figure 5: Problem G.2.

Solution 2. The quadrangle $ABCD$ with its vertices in the four points can be convex or non-convex.

If the quadrangle is non-convex, then one of the points lies in the interior of the triangle defined by the remaining three points (see Figure 5) — the circumcircle of that triangle has the required property.

Assume now that the quadrangle $ABCD$ is convex. Then it has a pair of opposite angles, the sum of which is at least 180° — assume these are at vertices B and D (see Figure 5). Suppose the ray from the circumcentre O of triangle ABC through point D intersects the circumcircle in D'. Since $\angle B + \angle D' = 180^\circ$ and $\angle B + \angle D' = 180^\circ$, the point D cannot be exterior to the circumcircle, which thus has the required property. □

Problem G.3 *Baltic Way 2007*

This problem seems forbidding at first glance, and one may vainly try to locate some special points on the lines possessing the desired property. But the ingenious solution is a non-constructive existence proof.

Problem. Let $t_1, t_2, \ldots, t_k$ be different straight lines in space, where $k > 1$. Prove that points P_i on t_i, for $i = 1, \ldots, k$, exist such that P_{i+1} is the projection of P_i on t_{i+1} for $i = 1, \ldots, k-1$, and P_1 is the projection of P_k on t_1.

Solution. If all lines are parallel to each other, choose any point P_1 on t_1 and project it onto t_i to find P_i, for $i = 2, \ldots, k$.

Otherwise, denote the acute or right angle between t_i and t_{i+1} by α_i, $i = 1, \ldots, k-1$, and the acute or right angle between t_k and t_1 by α_k. At least one of these angles is non-zero, so

$$\cos\alpha_1 \cos\alpha_2 \cdots \cos\alpha_k < 1.$$

Clearly, P_i, for $i = 2, \ldots, k$, is determined by a choice of P_1, so let P_{k+1} be the projection of P_k on t_1. We must prove that P_1 may be chosen such that P_1 and P_{k+1} coincide.

If P_1 moves along t_1 with constant speed v, then P_{k+1} moves along t_1 with constant speed

$$v \cdot \cos\alpha_1 \cos\alpha_2 \ldots \cos\alpha_k < v.$$

Hence, at some moment, P_1 and P_{k+1} coincide. □

Problem G.4 *Baltic Way 1993*

Again, this problem seems hopeless at first. But once again, no calculations are needed. The graceful proof is pure *idea*.

Problem. Let's consider three pairwise non-parallel straight lines in the plane. Three points are moving along these lines with different non-zero velocities, one on each line (we consider the movement to have taken place for infinite time and continue forever into the future). Is it possible to determine these straight lines, the velocities of each moving point and their positions at some "zero" moment in such a way that the points never were, are or will be collinear?

Solution. Yes, it is. First, place the three points at the vertices of an equilateral triangle at the "zero" moment and let them move with equal velocities along the straight lines determined by the sides of the triangle as shown in Figure 6.

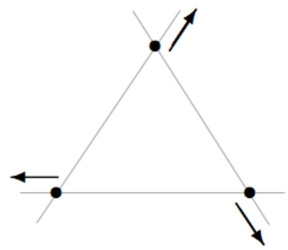

FIGURE 6: Problem G.4.

Then, at any moment in the past or future, the points are located at the vertices of some equilateral triangle, and thus cannot be collinear. Finally, to make the velocities of the points also differ, take any non-zero constant vector such that its projections on the three lines have different lengths and add it to each of the velocity vectors. This is equivalent to making the whole picture "drift" across the plane with constant velocity, so the non-collinearity of our points is preserved (in fact, they are still located at the vertices of an equilateral triangle at any given moment). □

Problem G.5 *Baltic Way 2006*

This problem is one of the most surprising in the geometry department. The formulation is extremely simple, and so is the solution, although the construction is hard to figure out. At first, one is amazed that such a construction is even possible.

Problem. There are 2006 points marked on the surface of a sphere. Prove that the surface can be cut into 2006 congruent pieces so that each piece contains exactly one of these points inside it.

Solution. Choose a North Pole and a South Pole so that no two points are on the same parallel of latitude and no point coincides with either pole. Draw parallels through each point. Divide each of these parallels into 2006 equal arcs so that no point is the endpoint of any arc. In the sequel, "to connect two points" means drawing the smallest arc of the great circle passing through these points. Denote the points of division by $A_{i,j}$, where i is the number of the parallel counting from North to South ($i = 1, 2, \ldots, 2006$), and $A_{i,1}$, $A_{i,2}$, $\ldots$, $A_{i,2006}$ are the points of division on the i'th parallel, where the numbering is chosen such that the marked point on the i'th parallel lies between $A_{i,i}$ and $A_{i,i+1}$.

Connect the points as follows:

$$\begin{gathered} N - A_{1,1} - A_{2,1} - A_{3,1} - \cdots - A_{2006,1} - S \\ N - A_{1,2} - A_{2,2} - A_{3,2} - \cdots - A_{2006,2} - S \\ \vdots \\ N - A_{1,2006} - A_{2,2006} - A_{3,2006} - \cdots - A_{2006,2006} - S \end{gathered}$$

These lines divide the surface of the sphere into 2006 parts which are congruent by rotation; each part contains one of the given points. □

Remark. It has been pointed out that the solution, as it stands, is erroneous. The poles of the sphere do not belong to any specific pieces of the decomposition, and any attempt to insert them would destroy the congruence property. If the boundaries of the pieces are disregarded, so that the 2006 pieces are only required to have congruent interiors, the solution works.

Number Theory

Problem N.1 *Baltic Way 2014*

The sublime beauty of the following problem lies in the meta-logic leading up to its solution.

(1) It is impossible to prove a number prime. Certainly, the number must be composite.

(2) Proving this requires exhibiting a prime factor, necessarily greater than 712. The next prime in succession is 719.

(3) The very form of the number cries out for an application of Wilson's Theorem.

Ah! how neatly the pieces of the puzzle come together!

Problem. Determine whether $712! + 1$ is a prime number.

Solution. Use Wilson's Theorem for the prime 719:

$$\begin{aligned}712! + 1 &\equiv 720 \cdot 712! + 1 = (-1) \cdot (-2) \cdot (-3) \cdot (-4) \cdot (-5) \cdot (-6) \cdot 712! + 1 \\ &\equiv 718 \cdot 717 \cdot 716 \cdot 715 \cdot 714 \cdot 713 \cdot 712! + 1 = 718! + 1 \equiv 0 \bmod 719. \quad \square\end{aligned}$$

Problem N.2 *Baltic Way 1999*

Many classical number-theoretical results find their use in the Baltic Way. Fermat's Little Theorem (and Euler's generalisation), Wilson's Theorem and the Chinese Remainder Theorem are among the most frequently cited. The problem below is exceptional in that the *Twin Prime Conjecture* provides a subtle hint to its solution. If no sequence existed as required in the problem, then, taking $c_1 = 0$ and $c_2 = 2$, there would be only finitely many integers a for which a and $a + 2$ were both prime — that is, there would be only finitely many twin primes. By a meta-argument, such a sequence *must necessarily exist*.

Problem. Does there exist a finite sequence of integers $c_1, \ldots, c_n$ such that all the numbers $a + c_1, \ldots, a + c_n$ are primes for more than one but not infinitely many different integers a?

Solution. Answer: yes.

Let $n = 5$ and consider the integers 0, 2, 8, 14, 26. Adding $a = 3$ or $a = 5$ to all of these integers we get primes. Since the numbers 0, 2, 8, 14 and 26 have pairwise different remainders modulo 5 then for any integer a the numbers $a + 0$, $a + 2$, $a + 8$, $a + 14$ and $a + 26$ have also pairwise different remainders modulo 5; therefore one of them is divisible by 5. Hence if the numbers $a + 0$, $a + 2$, $a + 8$, $a + 14$ and $a + 26$ are all primes then one of them must be equal to 5, which is only true for $a = 3$ and $a = 5$. □

Problem N.3 *Baltic Way 1998*

This is an example of a beautiful competition problem hiding deeper mathematics (and requiring heavier machinery) than appears at first glance. The formulation is cunning, requiring us to prove the *existence* of a prime factor of c greater than 5. It so transpires that *every* prime factor of c is greater than 5 (in an irreducible quasi-Pythagorean triple).

Problem. A triple of positive integers (a, b, c) is called *quasi-Pythagorean* if there exists a triangle with lengths of the sides a, b, c and the angle opposite to the side c equal to 120°. Prove that if (a, b, c) is a quasi-Pythagorean triple, then c has a prime divisor greater than 5.

Solution. By the Law of Cosines, a triple of positive integers (a, b, c) is quasi-Pythagorean if and only if

$$c^2 = a^2 + ab + b^2. \tag{6}$$

If a triple (a, b, c) with a common divisor $d > 1$ satisfies (6), then so does the reduced triple $(\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$. Hence it suffices to prove that in every irreducible quasi-Pythagorean triple the greatest term c has a prime divisor greater than 5. Actually, we will show that in that case every prime divisor of c is greater than 5.

Let (a, b, c) be an irreducible triple satisfying (6). Note that then a, b and c are pairwise coprime. We have to show that c is not divisible by 2, 3 or 5.

If c were even, then a and b (coprime to c) should be odd, and (6) would not hold.

Suppose now that c is divisible by 3, and rewrite (6) as

$$4c^2 = (a + 2b)^2 + 3a^2. \tag{7}$$

Then $a + 2b$ must be divisible by 3. Since a is coprime to c, the number $3a^2$ is not divisible by 9. This yields a contradiction since the remaining terms in (7) are divisible by 9.

Finally, suppose c is divisible by 5 (and hence a is not). Again we get a contradiction with (7) since the square of every integer is congruent to 0, 1 or -1 modulo 5; so $4c^2 - 3a^2 \equiv \pm 2 \bmod 5$ and it cannot be equal to $(a+2b)^2$. This completes the proof. □

Remark. A yet stronger claim is true: *If a and b are coprime, then every prime divisor $p > 3$ of $a^2 + ab + b^2$ is of the form $p = 6k + 1$.* (Hence every prime divisor of c in an irreducible quasi-Pythagorean triple (a, b, c) has such a form.)

This stronger claim can be proved by observing that p does not divide a and the number $g = (a + 2b)a^{\frac{p-3}{2}}$ is an integer whose square satisfies

$$g^2 = (a + 2b)^2 a^{p-3} = (4(a^2 + ab + b^2) - 3a^2)a^{p-3} \equiv -3a^{p-1} \equiv -3 \bmod p.$$

Hence -3 is a quadratic residue modulo p. This is known to be true only for primes of the form $6k + 1$; proofs can be found in many books on number theory, e.g. Ireland & Rosen: *A Classical Introduction to Modern Number Theory*, second edition, Springer-Verlag (1990).

Problem N.4 *Baltic Way 2009*

The most difficult part of this exquisite problem is probably deciding on the very first line of the proof.

Problem. Let a and b be integers such that the equation $x^3 - ax^2 - b = 0$ has three integer roots. Prove that $b = dk^2$, where d and k are integers and d divides a.

Solution. It is sufficient to prove, for each prime p, that, if b is divisible by p an odd number of times, then p divides a.

Let α, β, γ be the roots of the cubical equation. Then, by Viètes Formulæ,

$$\alpha + \beta + \gamma = a, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = 0 \qquad \text{and} \qquad \alpha\beta\gamma = b.$$

Since $p \mid b = \alpha\beta\gamma$, we may suppose $p \mid \alpha$. It follows from $\beta\gamma = -\alpha(\beta + \gamma)$ that β or γ is divisible by p; suppose it is β. If $p \nmid \gamma$, then α and β are divisible by the same power of p. Hence, $b = \alpha\beta\gamma$ contains an even number of factors p, contradicting our assumption. Hence $p \mid \gamma$, and so $p \mid \alpha + \beta + \gamma = a$. □

Problem N.5 *Baltic Way 2011*

There are two standard ways to prove that there are infinitely many positive integers with a given property. One is to explicitly construct them. The other is to give a recursive construction: to show, given one number with this property, how another may be constructed. The solution to this problem elegantly circumvents these procedures and takes another route altogether: proof by contradiction.

Problem. An integer $n \geqslant 1$ is called *balanced* if it has an even number of distinct prime divisors. Prove that there exist infinitely many positive integers n such that there are exactly two balanced numbers among n, $n+1$, $n+2$ and $n+3$.

Solution. We argue by contradiction. Choose N so large that no $n \geqslant N$ obeys this property. Now we partition all integers $\geqslant N$ into maximal blocks of consecutive numbers which are either all balanced or not. We delete the first block from the following considerations, now starting from $N' > N$. Clearly, by assumption, there cannot meet two blocks with length $\geqslant 2$. It is also impossible that two blocks meet of length 1 (remember that we deleted the first block). Thus all balanced or all unbalanced blocks have length 1. All other blocks have length 3, at least.

Case 1: All unbalanced blocks have length 1. We take an unbalanced number $u > 2N' + 3$ with $u \equiv 1 \bmod 4$ (for instance $u = p^2$ for an odd prime p). Since all balanced blocks have length $\geqslant 3$, $u - 3$, $u - 1$, and $u + 1$ must be balanced. This implies that $(u-3)/2$ is unbalanced, $(u-1)/2$ is balanced, and $(u+1)/2$ is again unbalanced. Thus $\{(u-1)/2\}$ is a balanced block of length 1 — contradiction.

Case 2: All balanced blocks have length 1. Now we take a balanced number $b > 2N' + 3$ with $b \equiv 1 \bmod 4$ (for instance $b = p^2q^2$ for distinct odd primes p, q). By similar arguments, $(b-3)/2$ is balanced, $(b-1)/2$ is unbalanced, and $(b+1)/2$ is again balanced. Now the balanced block $\{(b-1)/2\}$ gives the desired contradiction. $\square$